Fitting the Logistic Growth Model: Insights into Sustainable Fish Harvesting

This final report is submitted to the Department of Mathematics as partial fulfillment of the Bachelor of Science (Honors) Degree in Financial Mathematics and Industrial Statistics.

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Chapter 1

Introduction

1.1 Background of the Study

Due to awareness of the value of fish proteins and fats in human meals, the demand for fish has increased globally. Although most of the fish that humans eat come from sea fishing, the natural supply is inadequate to meet the growing population demand. Thus, inland aquaculture has enormous demand all over the world. In this context, the existence of approximately 12,000 non-perennial reservoirs in the dry zone (annual rainfall less than 187 cm) provides a significant opportunity for the growth of Sri Lanka's culture-based fisheries industry to meet the local demand for fish [6].

This paper highlights the potential of the aquaculture of tilapia in Sri Lanka, which can be produced under conditions of fresh, brackish, and salt water. In this paper, the logistic growth model has been used to study harvesting, and appropriate adaptation strategies must be identified to manage and increase productivity in the aquaculture sector. The presented approach attempts to find evidence for the application of the logistic population growth model in continuous time under conditions of continuous and optimal supply in aquaculture.

1.2 Significance of the Study

Tilapia is considered an important species in the freshwater fisheries of Sri Lanka, where a significant source of income is provided to local fishing communities and an important role is played in national food security. In this study, the impact of the density of the tilapia population and harvest rates is assessed using three models: constant, proportional, and periodic, with the use of a logistic growth model. Optimal harvest levels are determined for long-term development and reproduction of tilapia while maximizing economic benefits for fish producers.

1.3 Objective of the Study

The primary objective of this research was to use a logistic population growth model to identify optimal harvest levels that balance economic yield and environmental sustainability with a sustainable fisheries management framework for freshwater tilapia in Sri Lanka, thus exploring the efficiency and productivity of various sustainable fish harvesting strategies.

1.4 Problem Statement

This study addresses how much should be harvested using sustainable harvesting methods to balance environmental and economic sustainability.

Chapter 2

Literature Review

In aquaculture, fish harvesting models are crucial in optimizing economic sustainability and profitability. These models use mathematical frameworks to predict the most suitable harvesting proportion from the available population. The logistic growth model is the most widely recognized measure of the trend and behavior of species. Verhulst's early theoretical developments established the logistic equation, which has now been modified for several ecological uses [3].

The logistic growth model has been widely applied in fisheries across the world to control sustainable harvest levels. [4], for instance, examined periodic, proportional, and constant harvesting techniques and their effects on population stability. They showed that population extinction results from overharvesting that exceeds the limits. Despite its adaptability, proportional harvesting runs the danger of destabilizing when harvesting rates go close to population growth rates. [1] used comparable models to study tilapia fish populations in reservoirs in Albania. This study provided a useful framework for evaluating how various harvesting techniques affect fish populations, acting as a standard for other regions.

Sri Lanka's inland fisheries are crucial for food security and economic development, with over 12,000 non-perennial reservoirs in the dry zone providing ample opportunities for aquaculture. Tilapia, a key species, is adaptable to diverse water conditions, but research on optimizing harvesting using logistic growth models is limited. The literature on culture-based fishing in Sri Lanka lacks dynamic harvesting strategies, highlighting a critical gap [7].
The most important factor for the successful management of harvested populations is that harvesting strategies are sustainable. Harvesting has been considered a factor of stabilization, destabilization, improvement of mean population levels, induced fluctuations, and control of non-native predators [1].

Chapter 3

Methodology

3.0.1 Study Area

The study focuses on the **Udawalawe Reservoir**, a key inland aquaculture site managed by the *National Aquaculture Development Authority (NAQDA)*. The reservoir supports significant *GIFT tilapia* production.

Tilapia are hardy, relatively fast-growing fish and typically reach maturity between 3 and 6 months, depending on environmental factors such as water quality. It breeds multiple times per year with 100 to 1500 eggs from females. In addition, it has a lifespan of 5 to 10 years under natural conditions. In addition, in aquaculture, they are generally harvested between 6 and 12 months of age due to the identification of the economic and nutritional value of tilapia.

3.0.2 Data Collection

The data used in this study will be obtained from **NAQDA** and other relevant sources. The key data points include:

- Tilapia population trends over time (monthly/yearly records).
- Harvesting volumes under different strategies (constant, proportional, and periodic).
- Environmental parameters such as water temperature and quality.
- Stocking and replenishment efforts by NAQDA.

3.1 Research Approach

Quantitative analysis was the main focus. Differential equations were applied to derive these models because differential equations can represent a population's size as a function of time.

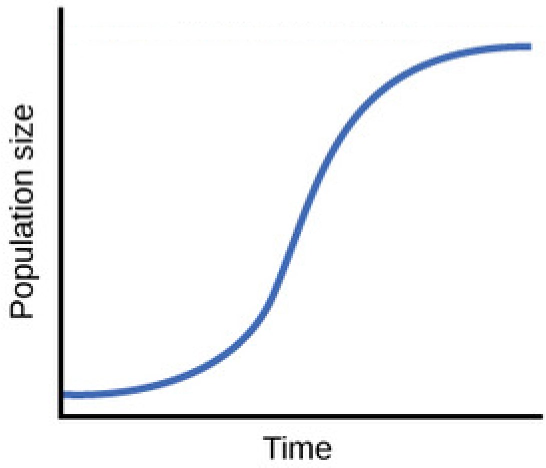
3.2 Research Design

In this case study, a logistic differential equation was applied to study population dynamics.

3.2.1 Population Growth and Carrying Capacity

The variable $\mathbf{t}$ represents time (the unit of time can be various according to the specified situation). $\mathbf{P}$ represents the variability of the population; therefore, $\mathbf{P(t)}$ is given the population as a function of time, and then the first derivative $\frac{\mathbf{dP}}{\mathbf{dt}}$ represents the instantaneous rate of change of the population.
An exponential growth function is $\mathbf{P}(\mathbf{t}) = \mathbf{P_0\, e^{rt}}$. In this function, $\mathbf{P(t)}$ represents the population at time $\mathbf{t}$, P_0 represents the initial population at time zero, and constant $\mathbf{r{>}0}$ is called the growth rate.

It verified that the function $\mathbf{P}(\mathbf{t}) = \mathbf{P_0\, e^{rt}}$ satisfies the initial-value problem $\frac{\mathbf{dP}}{\mathbf{dt}}$ with $\mathbf{P(0)} = P_0$.
However, this is unrealistic in a real-world setting because it is a prediction that, as time goes on, the population grows without bounds. Various factors limit a particular population's growth rate, including birth rate, death rate, food supply, predators, and so on.



The concept of carrying capacity **K** allows for the possibility that in a given area, only a certain number of a given organism or animal can thrive without running into resource issues.

3.2.2 Logistic Differential Equation

While **K** represents the carrying capacity for a particular organism in a given environment, let **r** be a real number that represents the growth rate. The function **P(t)** represents the population of this organism as a function of time, and the P_0 represents the initial population.

The logistic differential equation is given by:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{3.1}$$

3.2.3 Solving the Logistic Differential Equation

The logistic differential equation is given by:

$$
\begin{aligned}
\frac{dP}{P(1-\frac{P}{K})} &= rdt \\
\int \frac{dP}{P(1-\frac{P}{K})} &= \int rdt \\
\int \left(\frac{1}{P} + \frac{1}{K-P}\right) dP &= rt + C \\
\ln|P| - \ln|K-P| &= rt + C \\
\ln\left|\frac{P}{K-P}\right| &= rt + C \\
\frac{P}{K-P} &= Ae^{rt} \quad (\text{where } A = e^{C}) \\
P &= KAe^{rt} - PAe^{rt} \\
P(1 + Ae^{rt}) &= KAe^{rt} \\
P(t) &= \frac{KAe^{rt}}{1 + Ae^{rt}}
\end{aligned}
$$

To determine the constant A, the initial condition is $P(0) = P_0$:

$$
\begin{aligned}
P_0 &= \frac{KA}{1+A} \\
P_0 + AP_0 &= KA \\
A &= \frac{P_0}{K - P_0}
\end{aligned}
$$

$$
P(t) = \frac{KP_0e^{rt}}{K + P_0(e^{rt} - 1)} \tag{3.2}
$$

3.2.4 Mathematical Models of Logistic Growth Population Equation

Logistic Growth Without Harvesting

$$
\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)
$$

$P(t)$ is the population time t, r the growth rate, and K the carrying capacity. The equilibrium points of equation (4) are $P = 0$ and $P = K$. The change of $\frac{dP}{dt}$ is zero if $P = 0$ or $P = M$.

If the positive initial population is smaller than the carrying capacity K, then the population density $P(t)$ increases. If the initial population is greater than the carrying capacity K, then it will decrease K monotonically. Indeed, the behavior of the solution for the first case is logistic growth.

For very small values of P, it is almost exponential, while $P > \frac{K}{2}$, it is asymptotically close to the constant value K, describing the carrying capacity of the environment. Some of the related models are discussed [8].

Constant Harvesting

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - h, \quad P(0) = p_0 \tag{3.3}$$

where h is a fixed amount of fish is harvested monthly.
The fixed points, P^*, are the solutions of the equation:

$$rP^*\left(1 - \frac{P^*}{K}\right) = h$$

The model has two fixed points:

$$P^*_{1,2} = \frac{1}{2}\left(K \pm \sqrt{M^2 - \frac{4hK}{r}}\right)$$

if $0 < h < \frac{rK}{4}$; one fixed point $P^* = \frac{K}{2}$ when $h = \frac{rK}{4}$; no fixed point when $h > \frac{rK}{4}$.

Proportional Harvesting

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - hP, \quad P(0) = P_0 \tag{3.4}$$

h is the harvest proportion, and K is the carrying capacity with no harvesting. The fixed points, P^*, are the solutions of the equation:

$$rP^*\left(1 - \frac{P^*}{M}\right) = hP^*$$

That is,

$$P^* = 0$$

and

$$P^* = \frac{(r-h)M}{r}$$

The extinction fixed point, $P^* = 0$, is unstable for values of $h < r$. As h increases, the larger equilibrium (carrying capacity) shrinks, but it remains stable for $h < r$.

Periodic Harvesting

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - h\left(1 + \sin\left(\frac{2\pi t}{12}\right)\right) \tag{3.5}$$

This model simulates seasonal variations, alternating between periods of harvesting and rest, where h the harvesting rate is defined as: $h(t) = h_0(1 + \sin(2\pi t))$, where $(1 + \sin(2\pi t))$ denotes the periodicity or frequency of fishing cycles per unit of time.

3.2.5 Assumptions of the logistic model

- There is a constant carrying capacity, although there are some fluctuations in the real-world scenarios.

- There is no immigration or emigration; the only factor affecting it is the harvesting.
- Homogeneous population, which identically affects reproduction and uses resources

Chapter 4

Data Analysis

4.1 Descriptive Analysis

The center of the Udawalawe reservoir is 06 ha in area and consists of broodstock ponds, rearing and nursery ponds, and hatchery facilities [2].

Fish Rearing Facilities in Udawalawa Tilapia AQDC	**No. of ponds**	**Area (m^2)**
Broodstock ponds	09	4990
Rearing ponds	02	450
Nursery ponds (Cemented cubicles)	34	920
Total	**45**	**6360**

Table 4.1: Summary of Fish Rearing Facilities in Udawalawa Tilapia AQDC

Month	**2021**	**2022**	**2023**	**2024**
January	20075	47159.3	29478.4	26432.7
February	16907	35289.9	36235.1	15992.1
March	13629	31797.4	31872.8	11000.6
April	24623	44352.2	44377.4	20327.5
May	33424	25124.2	24338.8	9111.6
June	9419	33778.7	33725.3	10233.6
July	7287	55779.4	57414.1	22763.1
August	7012	84717.5	87737.5	25507.4
September	14055	69767.4	71167.0	29000.0
October	10415	70458.1	72134.7	25668.0
November	17285	71235.0	72490.2	32070.9
December	19318	48489.1	48375.7	15467.8
Total	203479	617948.8	589346.5	268575.3

Table 4.2: Actual and Forecasted Tilapia Production (Mt) for 2021–2024

Based on this information derived from the trend analysis, it was observed that November, December, and January consistently exhibit the maximum level of tilapia produc-

tion. In contrast, June, July, and August were associated with the minimum level of tilapia production [5].
Since the expected data could not be obtained (pH value of water, water temperature, supplied food quality, and other seasonal effects to calculate the exact population growth rate), this case study has assumed the population growth rate is 0.8. carrying capacity is 2,000,000 [5].

$$\frac{dP}{dt} = 0$$
$$rP\left(1 - \frac{P}{K}\right) = 0$$
$$0.8P\left(1 - \frac{P}{2000000}\right) = 0$$
$$0.8P = 0 \quad \text{or} \quad 1 - \frac{P}{2000000} = 0$$
$$P = 0 \quad \text{or} \quad P = 2,000,000$$

Thus, the equilibrium points are $P = 0$ and $P = 2,000,000$.

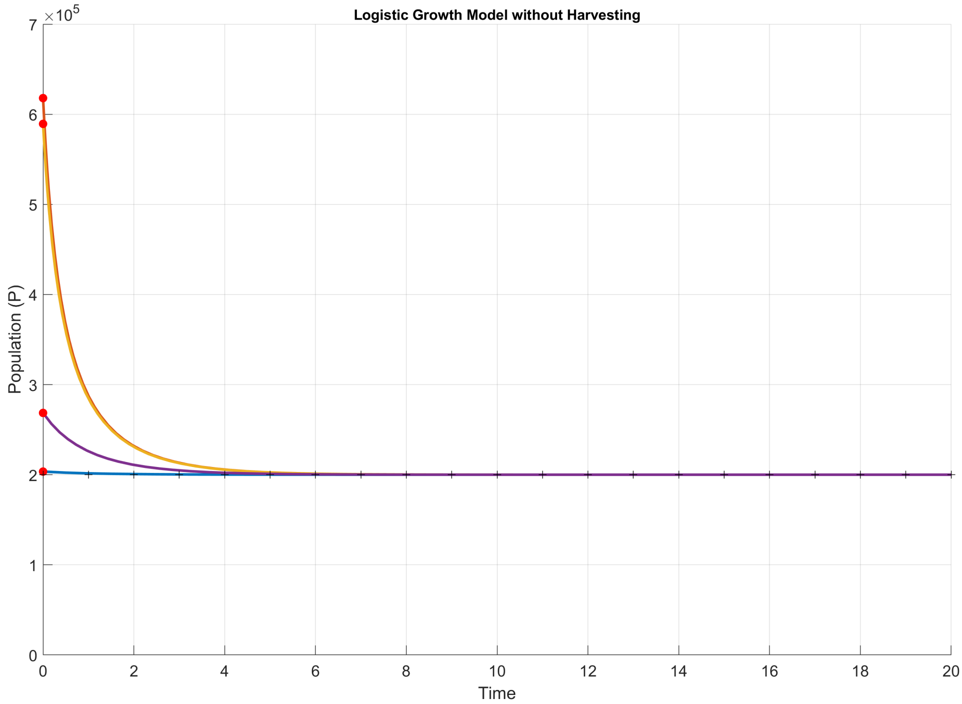


Figure 4.1: Logistic Growth Model without Harvesting

This means that if the initial population starts at P = 0, it will remain at P = 0. Similarly, if the initial population starts at P = 2,000,000, it will remain at the same level. The stability of these equilibrium points can be seen in Figure 4.1, and an equilibrium point has been obtained according to the available data.

4.2 Exploratory Analysis

As harvesting increases, the two fixed points move closer to each other, with the lower fixed point remaining unstable and the upper fixed point remaining stable [1].

Logistic Growth Model with Constant Harvesting

In this method, it is considered that only one fixed harvest is obtained throughout the year.

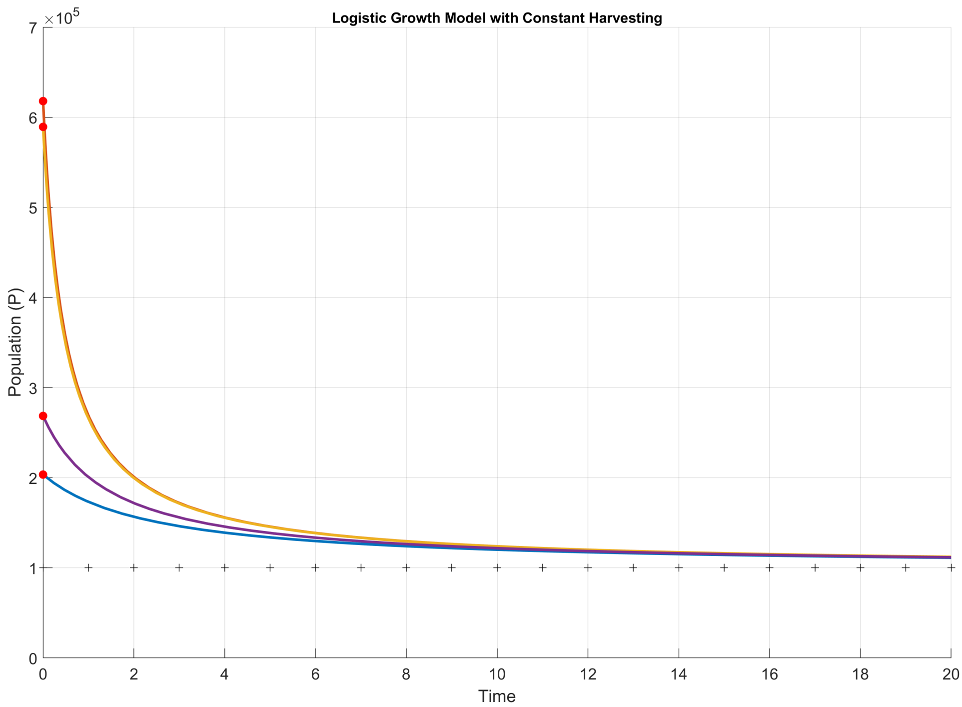


Figure 4.2: Logistic Growth Model with Constant Harvesting ($h = h_{max}$)

For the maximum harvest amount $h = 40,000$, Figure 4.2 shows an equilibrium point near half of the carrying capacity. For P_0 larger than 1,000,000, the population will decrease and approach 1,000,000. Theoretically, for less than 1,000,000, the population will lead to extinction.

Figure 4.3 shows the decreasing trends of the tilapia population when harvest amounts exceed the maximum harvest. This indicates that the fish population will go to extinction notwithstanding the initial population size. This is to say that overfishing during one year can result in a sudden drop in fish tilapia in subsequent years. Hence, fish farmers must be cautious and not exceed 2,000,000 in fishing quotas.

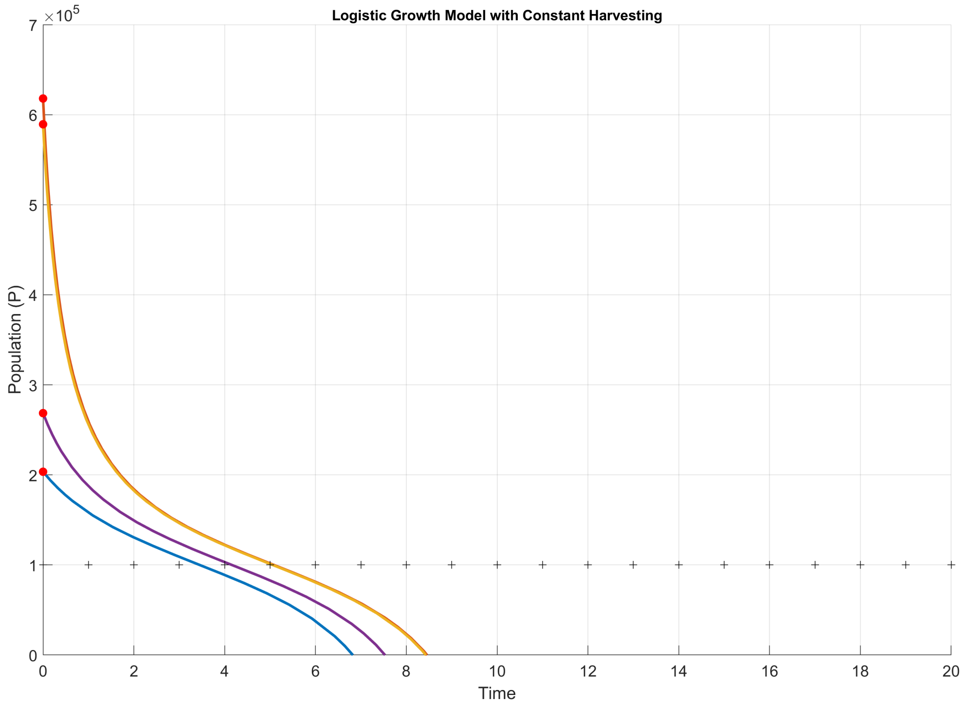


Figure 4.3: Logistic Growth Model with Constant Harvesting ($h > h_{max}$)

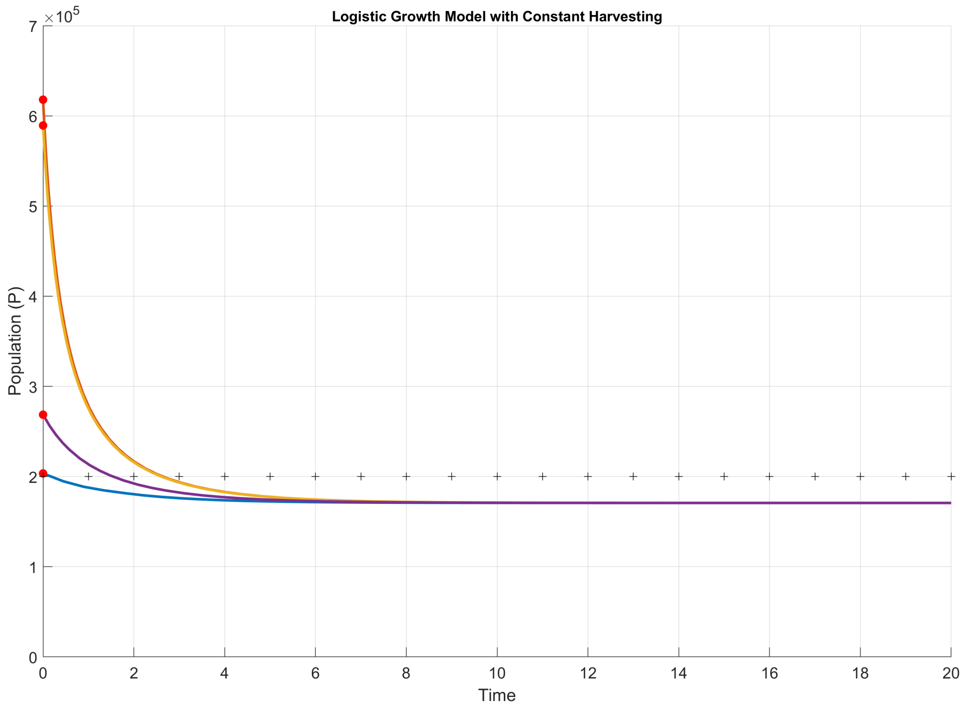


Figure 4.4: Logistic Growth Model with Constant Harvesting ($h < h_{max}$)

For $h < 40,000$, theoretically there should be two equilibrium points that exist when the value of harvesting is less than 2,000,000. According to Figure 4.4, the upper equilibrium point is slightly less than 2,000,000.

Logistic Growth Model with Proportional Harvesting

In this method, the relationship between the harvesting rate h and growth rate r is examined. In addition, harvesting is a proportion of the population growth rate.

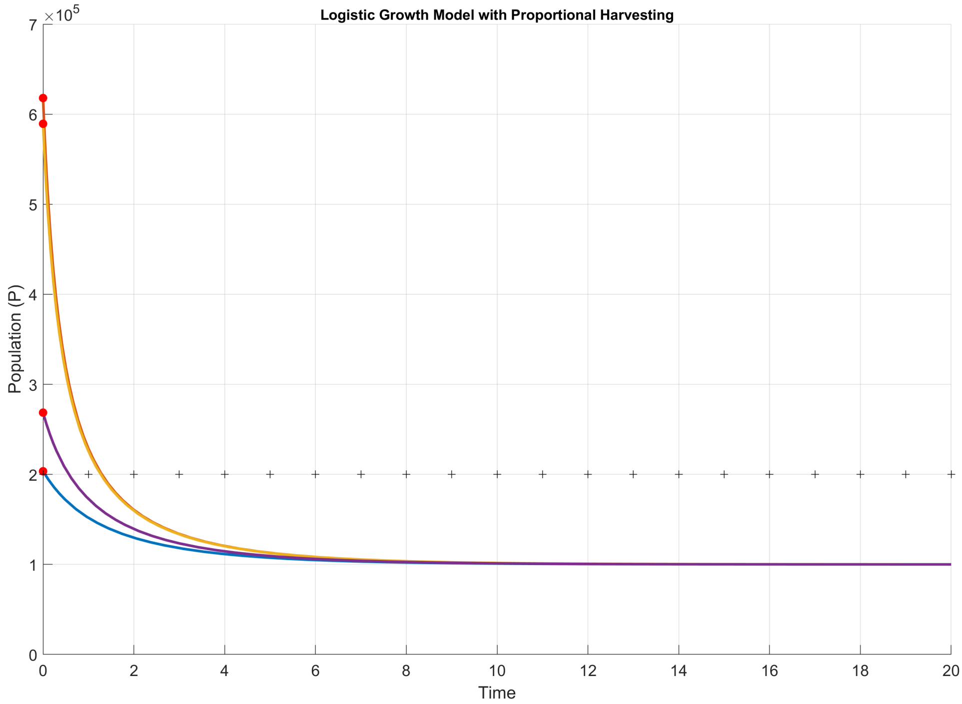


Figure 4.5: Logistic Growth Model with Proportional Harvesting ($h = r/2$)

Figure 4.5 shows the situation in which the harvest rate and growth rate are equal. According to the results, the time to approach half of the proportion is relatively less than the time to approach half of the population in the method of constant harvesting when the harvesting amount is equal to its maximum value.

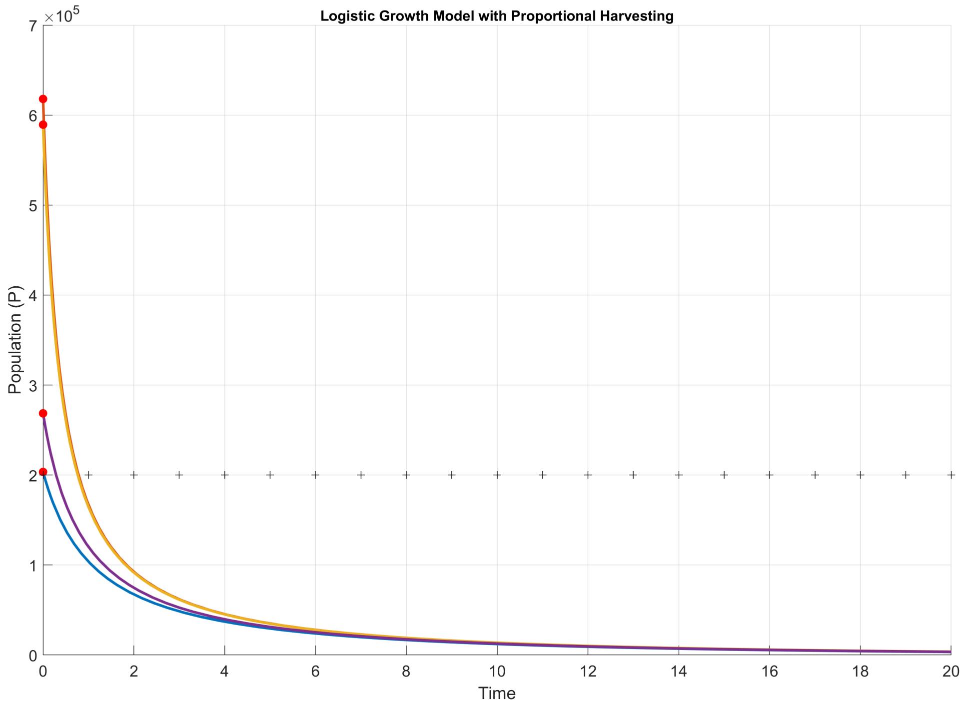


Figure 4.6: Logistic Growth Model with Proportional Harvesting ($h > r$)

As shown in Figure 4.6, the tilapia population goes extinct when the harvest rate is greater than the growth rate. When the harvest rate r increases, the slope of the curve increases, which implies that overfishing during one year can potentially extirpate the fish in the reservoir. That is why it is crucial not to exceed the fishing quotas.

Based on Figure 4.7 below, it can be depicted that when the harvesting rate is less than the population growth rate, the fish population becomes stable over time, which is lower than the carrying capacity level, and it is a relatively lower amount than the situation of constant harvesting with the harvest that is less than its maximum harvest.

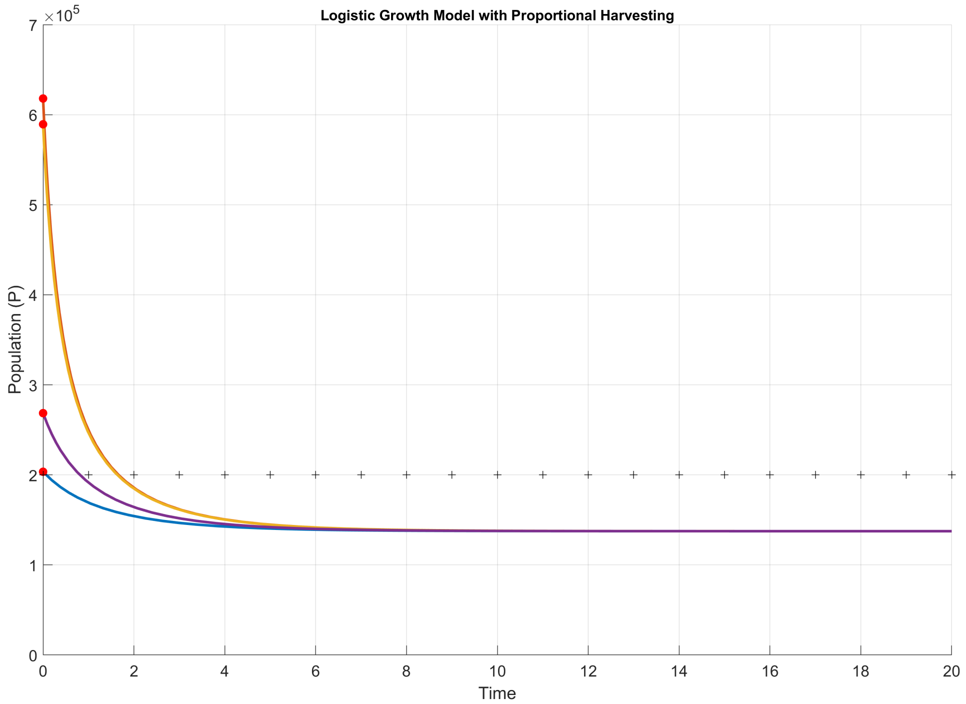


Figure 4.7: Logistic Growth Model with Proportional Harvesting ($h < r$)

Logistic Growth Model with Periodic Harvesting

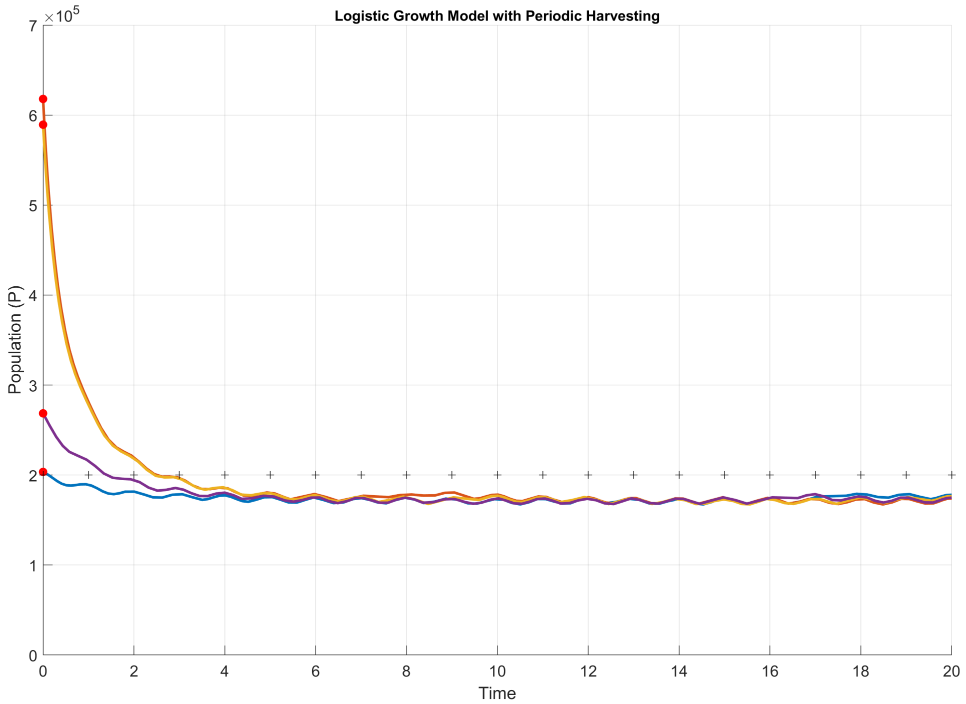


Figure 4.8: Logistic Growth Model with Periodic Harvesting ($h < h_{max}$)

The population fluctuates from year to year but remains stable, demonstrating resilience. The ponds have a total initial population of 2,000,000 tilapia fish. During the first year, it is assumed that the amount of tilapia harvest is below its maximum harvest amount, allowing a proportion of tilapia fish to grow without being harvested for a year, and this pattern is repeated for several years. Theoretically, the tilapia population should be increased until carrying capacity is reached when there is no harvest during the year, but in this case, it is below carrying capacity (Figure 4.8).

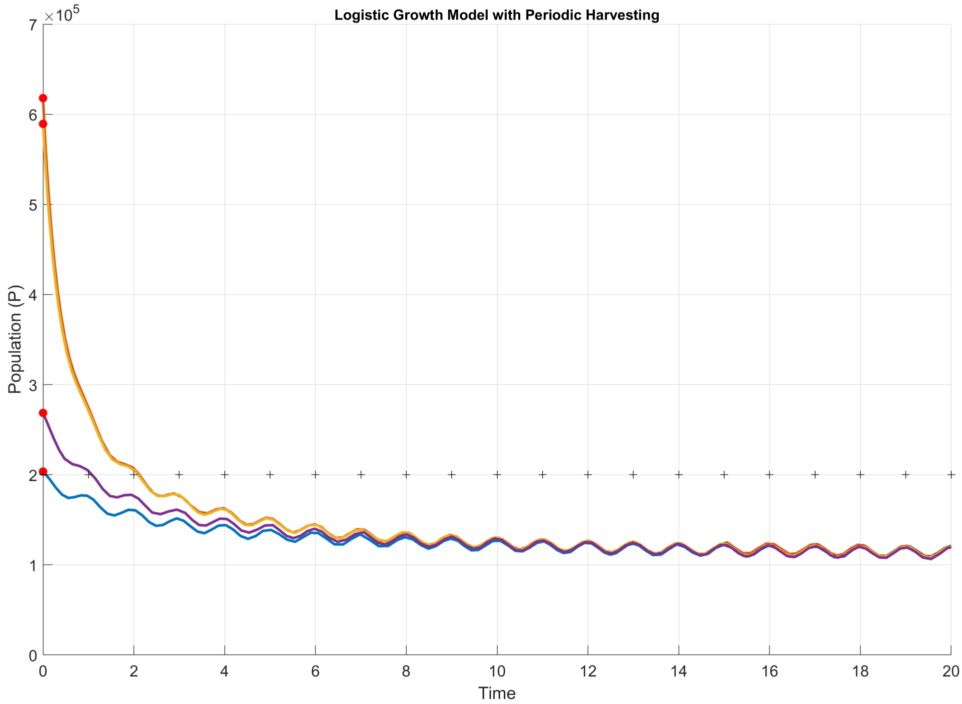


Figure 4.9: Logistic Growth Model with Periodic Harvesting ($h = h_{max}$)

The population dynamics of tilapia under periodic harvesting at the maximum sustainable yield are illustrated in Figure 4.9. A sharp decline from a high starting value was initially experienced by the population due to the impact of harvesting. Over time, the population is stabilized near a consistent value of approximately half the carrying capacity, representing a sustainable balance where the natural growth capacity aligns with the harvesting rate. Initial fluctuations in the population are caused by the periodic harvesting strategy, which gradually diminishes, indicating the resilience and adaptation of the system to the applied harvesting pattern. It is demonstrated by this outcome that long-term population stability can be ensured through periodic harvesting at maximum harvesting amounts while supporting sustainable harvests, provided favorable environmental conditions are maintained.

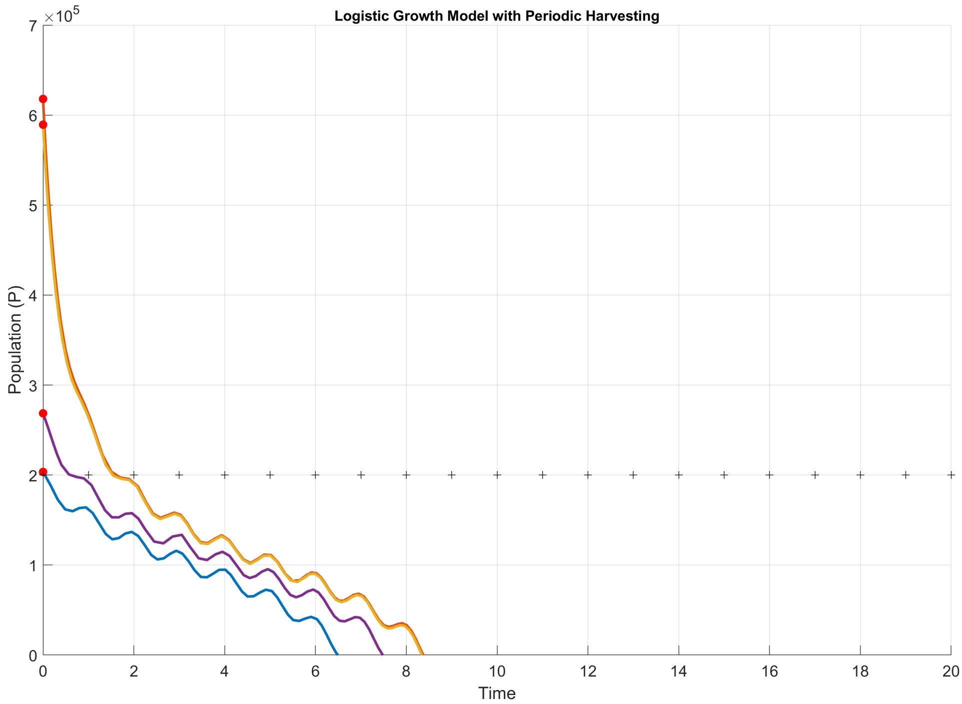


Figure 4.10: Logistic Growth Model with Periodic Harvesting ($h > h_{max}$)

The dynamics of a logistic growth model subjected to periodic harvesting is illustrated in this graph, where the maximum sustainable threshold is exceeded by the harvesting rate. A rapid decline is experienced by the population over time, eventually approaching extinction. Harvesting intervals are reflected by periodic dips in the graph, leading to a consistent reduction in the population size. This behavior shows that population collapse results from exceeding the sustainable harvesting rate, highlighting the critical importance of maintaining harvesting rates within sustainable limits.

Chapter 5

Discussion and Conclusion

Fish supply cannot be solely dependent on marine/sea fisheries, and alternatives must be found in the commercialization of aquaculture. Developing appropriate fish harvesting strategies can help meet market demand. In the case of a continuous harvesting strategy, when the value of the fish harvest exceeds its maximum value, it quickly becomes exhausted without recovering. In cases where a fixed harvesting strategy must be used, it is best to harvest enough to achieve the maximum harvest possible with less work. This allows the fish population to recover relatively quickly. The possibility of obtaining a fish harvest above the maximum that can be obtained with the proportional harvesting strategy is relatively higher compared to the fixed harvesting strategy. Also, the proportional harvesting strategy quickly brings the fish population to equilibrium. With the results of the periodic harvesting strategy, we can conclude that it is the most economical and environmentally efficient method. However, it depends on the time of use. All things considered, this third method is the most appropriate. The solution to optimize the harvest while maintaining the fish population is the seasonal harvesting strategy. To improve productivity, shorten the return on investment period, and minimize the risk of changes in the selling price and production costs of the products, a harvesting strategy using seasonal harvesting strategies can be used, especially when using comparatively short return periods. However, with constant harvesting, fish farming does not have enough time to restore the fish population. By developing a fish harvesting strategy, market demand can be met. Farmers can improve commercial returns before harvesting. This study helps the farmer to establish freshwater ponds such as tilapia fish, like any other agricultural activity.

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Appendix

```
M = 200000;         % Carrying capacity
r = 0.8;          % Growth rate
h = 60000;           % Constant harvesting rate (0-40000)
tspan = [0 20];   % Time range from 0 to 20
max_harvest = r*M/4;

% logistic equation
logistic_harvesting = @(t, P) r * P * (1 - P / M) - h*(1+sin(2*pi*t));

P0_values = [203479, 617948.8, 589346.5, 268575.3];

figure('Position', [100, 100, 1200, 800]);
hold on;

for i = 1:length(P0_values)
    P0 = P0_values(i);
    [t, P] = ode45(logistic_harvesting, tspan, P0);
    plot(t, P, 'LineWidth', 2);
    plot(0, P0, 'ro', 'MarkerFaceColor', 'r');
    plot(0:20,200000, 'k+', 'MarkerFaceColor', 'r');
end

xlabel('Time');
ylabel('Population (P)');
title('Logistic Growth Model with Periodic Harvesting');
grid on;
ylim([0 700000]);
xlim([0 20]);

xticks(0:2:20)
yticks(0:100000:700000);
ytickformat('%d');

% Bold specific y-axis values (P0 values)
ax = gca;
ax.YAxis.FontSize = 12;
ax.XAxis.FontSize = 12;
for i = 1:length(P0_values)
    idx = find(ax.YTick == P0_values(i));
    if ~isempty(idx)
        ax.YTickLabel{idx} = ['\bf', ax.YTickLabel{idx}];
    end
end
hold off;

output_file = 'C:\Users\Use\Desktop\logistic_growth_model9.png';
print(output_file, '-dpng', '-r300');
```